Weighting

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Linear Methods in Causal Inference
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Review

- Under strong ignorability, we can use matching to account for the influence of confounders.
- We can directly match on the confounders (NN matching) or on the estimated propensity score (PS matching).
- NN matching is biased when there are more than one continuous confounders.
- But bias correction estimators exist and the bias is negligible under certain conditions.
- Classical bootstrap does not work for NN matching, but wild bootstrap does.

Weighting

- Another choice is to estimate the propensity scores and use either the HT or the HA estimator.
- In observational studies, they are named as the inverse probability of treatment weighting (IPW) estimator and the stabilized IPW estimator.
- Since the propensity scores are estimated, neither estimator is unbiased.
- As long as the propensity scores are consistently estimated and converging at a fast rate, both are consistent and asymptotically normal.
- The estimation of the propensity scores introduces extra uncertainties to the ATE estimate.

Weighting: estimation

- Remember that the HT and HA estimators take the form of:

$$\hat{\tau}_{HT} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{D_i Y_i}{\hat{g}(\mathbf{X}_i)} - \frac{(1-D_i)Y_i}{1-\hat{g}(\mathbf{X}_i)}\right),$$

$$\hat{\tau}_{HA} = \frac{\sum_{i=1}^{N} D_i Y_i/\hat{g}(\mathbf{X}_i)}{\sum_{i=1}^{N} D_i/\hat{g}(\mathbf{X}_i)} - \frac{\sum_{i=1}^{N}(1-D_i)Y_i/(1-\hat{g}(\mathbf{X}_i))}{\sum_{i=1}^{N}(1-D_i)/(1-\hat{g}(\mathbf{X}_i))}.$$

- We can estimate the ATT or ATC using similar ideas:

$$\hat{\tau}_{HA,ATT} = \frac{\sum_{i=1}^{N} D_i Y_i}{\sum_{i=1}^{N} D_i} - \frac{\sum_{i=1}^{N}(1-D_i)\hat{g}(\mathbf{X}_i)Y_i/(1-\hat{g}(\mathbf{X}_i))}{\sum_{i=1}^{N}(1-D_i)\hat{g}(\mathbf{X}_i)/(1-\hat{g}(\mathbf{X}_i))}.$$

Weighting: estimation

- To implement the Hajek estimator for the ATE, we first estimate the propensity score (using logistic regression) and obtain $\hat{g}(\mathbf{X}_i)$.
- Next, we construct the weight,

$$W_i = \frac{D_i}{\hat{g}(\mathbf{X}_i)} + \frac{1 - D_i}{1 - \hat{g}(\mathbf{X}_i)}.$$

- Finally, we regress Y_i on D_i and weight each unit by W_i.
- For the Hajek estimator for the ATT,

$$W_i = D_i + \frac{(1 - D_i)\hat{g}(\mathbf{X}_i)}{1 - \hat{g}(\mathbf{X}_i)}.$$

Weighting: inference

- If $g(\mathbf{X}_i)$ is known, we can just use the HC2 variance estimator in regression.
- But $g(\mathbf{X}_i)$ is estimated.
- We can write

$$\hat{\tau} - \tau = \hat{\tau} - \hat{\tau}(g) + \hat{\tau}(g) - \tau,$$

where $\hat{\tau}(g)$ is the "oracle estimator:"

$$\hat{\tau}_{HT}(g) = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{D_i Y_i}{g(\mathbf{X}_i)} - \frac{(1 - D_i) Y_i}{1 - g(\mathbf{X}_i)} \right).$$

- Then, we have

$$\begin{aligned} Var[\hat{\tau} - \tau] = & Var[\hat{\tau}(g) - \tau] + Var[\hat{\tau} - \hat{\tau}(g)] \\ & - 2Cov[\hat{\tau} - \hat{\tau}(g), \hat{\tau}(g) - \tau]. \end{aligned}$$

Weighting: inference

- If we conduct inference assuming $g(\mathbf{X}_i)$ is known, then the second and the third term in the expression are ignored.
- However, we can prove that

$$Var[\hat{\tau} - \tau] \leq Var[\hat{\tau}(g) - \tau],$$

  hence the variance will be conservative if we ignore the extra terms.
- For the HT estimator, if the logistic regression model is correctly specified, then the extra terms equal to

$$-E\left[\frac{(g(\mathbf{X}_i)m_0(\mathbf{X}_i) + (1 - g(\mathbf{X}_i))m_1(\mathbf{X}_i))^2}{g(\mathbf{X}_i)(1 - g(\mathbf{X}_i))}\right] \leq 0.$$

Weighting: inference

- The result implies that using the estimated propensity scores is more efficient than using the true propensity scores (suppose we know it)!
- In the Bernoulli trial, the propensity score is p for any unit and the estimated propensity score is $\frac{\sum_{i=1}^{N} D_i}{N}$.
- Therefore, the IPW estimator using the true propensity scores is the Horvitz-Thompson estimator, while the one using the estimated propensity scores is the Hajek estimator.
- We already know that the latter is more efficient!
- In observational studies, this conclusion was first proved by Hirano, Imbens, and Ridder (2003).
- Intuitively, estimating the propensity scores extracts more information from data.

Weighting: inference

- The IPW estimator using the estimated propensity scores reaches the efficiency bound derived by Hahn (1998).
- Let's define $m_1(\mathbf{X}_i) = E[Y_i(1)|\mathbf{X}_i]$, $m_0(\mathbf{X}_i) = E[Y_i(0)|\mathbf{X}_i]$, $\sigma_1^2(\mathbf{X}_i) = Var[Y_i(1)|\mathbf{X}_i]$, and $\sigma_0^2(\mathbf{X}_i) = Var[Y_i(0)|\mathbf{X}_i]$, then the bound is

$$E\left[\frac{\sigma_1^2(\mathbf{X}_i)}{g(\mathbf{X}_i)} + \frac{\sigma_0^2(\mathbf{X}_i)}{1 - g(\mathbf{X}_i)} + (m_1(\mathbf{X}_i) - m_0(\mathbf{X}_i) - \tau)^2\right].$$

- No estimator under strong ignorability can do better than this.

Weighting: pros and cons

- Weighting does not require bias correction or drop any units.
- But we need to have accurate predictions for the propensity score.
- There is a trade-off between convergence rate and accuracy.
- In practice, the estimated propensity score can be very close to 0 or 1.
- It is caused by the failure of positivity.
- Then, the HT estimator will have a huge variance.
- The HA estimator performs better in this case.
- One choice to trim units whose propensity score takes extreme values (Ma and Wang 2020).
- It alters the estimand and causes bias.
- Another choice is to use the covariate balancing propensity scores (Imai and Ratkovic 2014).

Covariate balancing propensity scores

- We have proved that the propensity score is a balance score:

$$D_i \perp \mathbf{X}_i | g(\mathbf{X}_i).$$

- We can exploit this property to improve the accuracy of our estimation.
- For any function of the covariates, $f(\mathbf{X}_i)$, we should have:

$$E\left[\frac{D_i f(\mathbf{X}_i)}{g(\mathbf{X}_i)} - \frac{(1-D_i) f(\mathbf{X}_i)}{1-g(\mathbf{X}_i)}\right] = 0.$$

- In finite sample, we should expect

$$\sum_{i=1}^{N}\left[\frac{D_i f(\mathbf{X}_i)}{g(\mathbf{X}_i)} - \frac{(1-D_i) f(\mathbf{X}_i)}{1-g(\mathbf{X}_i)}\right] \approx 0.$$

- We can set $f(\mathbf{X}_i)$ to be each of the covariates or their higher order terms.

Covariate balancing propensity scores

- Remember that we often estimate the propensity score via the logistic model:
$$g(\mathbf{X}_i) = \frac{e^{\mathbf{X}_i\beta}}{1 + e^{\mathbf{X}_i\beta}}.$$
- The first order condition is
$$\sum_{i=1}^{N} \left[\frac{D_i g'(\mathbf{X}_i)}{g(\mathbf{X}_i)} - \frac{(1 - D_i) g'(\mathbf{X}_i)}{1 - g(\mathbf{X}_i)} \right] = 0.$$
- Therefore, the same balance condition holds for $g'(\mathbf{X}_i)$ as well.
- We can combine all these balance conditions to estimate propensity scores more precisely.

Covariate balancing propensity scores

- We try to find a set of values, $\{\hat{p}_i\}_{i=1}^N = \{\hat{g}(\mathbf{X}_i)\}_{i=1}^N$, such that all the balance conditions are satisfied.
- In logistic regression, we have only one balance condition.
- Estimation could be done by the General Methods of Moments (Hansen 1982).
- Suppose we have K balance conditions: $\Psi(\mathbf{p}) = (\Psi_1(\mathbf{p}), \Psi_2(\mathbf{p}), \ldots, \Psi_K(\mathbf{p}))$, where $\mathbf{p} = (p_1, p_2, \ldots, p_N)$.
- We try the following problem

$$\{\hat{p}_i\}_{i=1}^N = \arg\min_{\mathbf{p}} \hat{E}[\Psi]' \widehat{Var}^{-1}[\Psi] \hat{E}[\Psi]$$

- We then rely on these $\{\hat{p}_i\}_{i=1}^N$ to construct the IPW estimators.

Covariate balancing propensity scores

- CBPS can handle continuous treatment variables (Fong, Hazlett, and Imai 2018).
- We find weights that are orthogonal to $\mathbf{X}$, D, and their interaction

$$\sum_{i=1}^{N} p_i \mathbf{X}_i = 0, \sum_{i=1}^{N} p_i D_i = 0$$

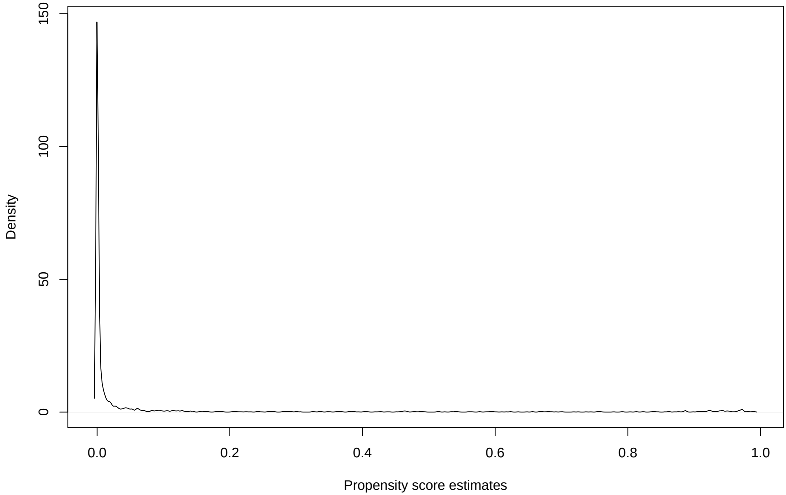
$$\sum_{i=1}^{N} p_i (\mathbf{X}_i * D_i) = 0, \sum_{i=1}^{N} p_i = N$$

- The properties of CBPS are derived in Fan et al. (2016).
- CBPS forces the propensity scores to balance the covariates, hence the estimates are less likely to be extreme.

Weighting: application

```
## The OLS estimate is 1794.343
## The SE of OLS estimate is 670.9967
## The Lin regression estimate is 1583.468
## The SE of Lin regression estimate is 678.0574
```

Weighting: application



Weighting: application

```
## The IPW ATT estimate is 2796.213

## The SE of IPW ATT estimate is 862.6273

##              mean.Tr  mean.Co   sdiff T pval
## age           25.816   23.812  28.012  0.002
## education     10.346   10.286   2.977  0.772
## black          0.843    0.818   6.853  0.483
## hispanic       0.059    0.120 -25.665  0.021
## married        0.189    0.093  24.449  0.005
## nodegree       0.708    0.716  -1.698  0.858
## re74        2095.574 1434.631  13.526  0.130
## re75        1532.056 1344.515   5.826  0.511
## u74            0.708    0.812 -22.795  0.012
## u75            0.600    0.413  38.134  0.000
```

Weighting: application

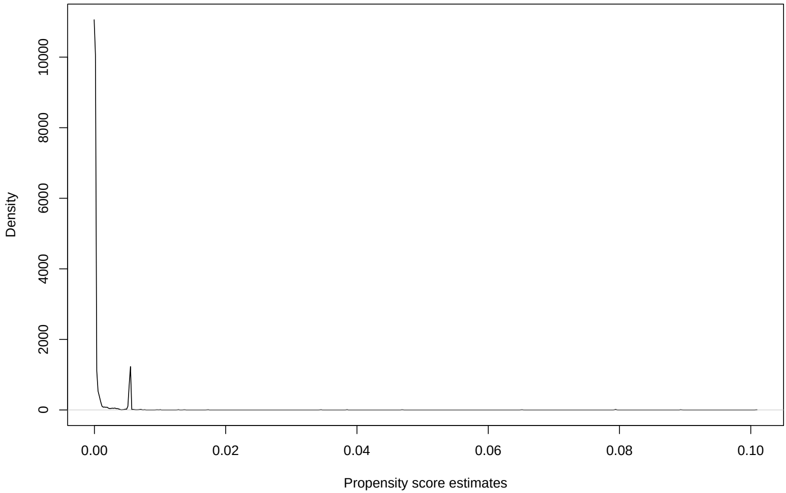
```
## The IPW ATT estimate is 1767.74

## The SE of IPW ATT estimate is 1116.721

##              mean.Tr  mean.Co   sdiff T pval
## age           27.021   25.382  23.171  0.053
## education     10.479   10.844 -17.496  0.136
## black          0.826    0.886 -15.679  0.151
## hispanic       0.069    0.025  17.252  0.080
## married        0.188    0.131  14.349  0.194
## nodegree       0.674    0.691  -3.689  0.753
## re74        1900.917 2186.548  -7.085  0.580
## re75        1204.959 1682.989 -23.763  0.107
## u74            0.681    0.688  -1.492  0.899
## u75            0.604    0.635  -6.182  0.598
```

Weighting: application

```
## [1] "Finding ATT with T=1 as the treatment.  Set ATT=2 t
```



Weighting: application

```
## The IPW ATT estimate is 2437.704

## The SE of IPW ATT estimate is 896.333

##              mean.Tr  mean.Co T pval
## age           25.816   25.889      1
## education     10.346   10.330      1
## black          0.843    0.838      1
## hispanic       0.059    0.061      1
## married        0.189    0.192      1
## nodegree       0.708    0.707      1
## re74        2095.574 2134.026      1
## re75        1532.056 1555.604      1
## u74            0.708    0.707      1
## u75            0.600    0.616      1
```

References I

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