Integration

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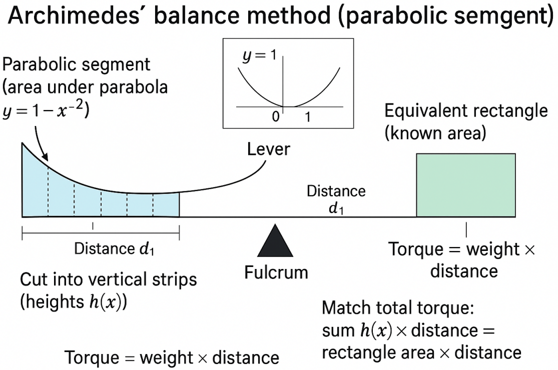
Motivations

- How can we calculate the area of an arbitrary shape?
- The first successful attempt was made by Archimedes.
- He calculated the area under a parabola using a method based on physics.
- Assume the shape is made of metal with density ρ.
- Place this piece of metal on one side of a lever.
- To balance its weight, put a rectangular piece of metal on the other side.
- How large does this rectangle need to be?
- We can determine it using the principle of the lever.

Motivations

- Cut the shape into small pieces.
- For each piece, record its weight and its distance from the lever's fulcrum.
- Enlarge the rectangle so that its torque matches the combined torque of all the pieces.

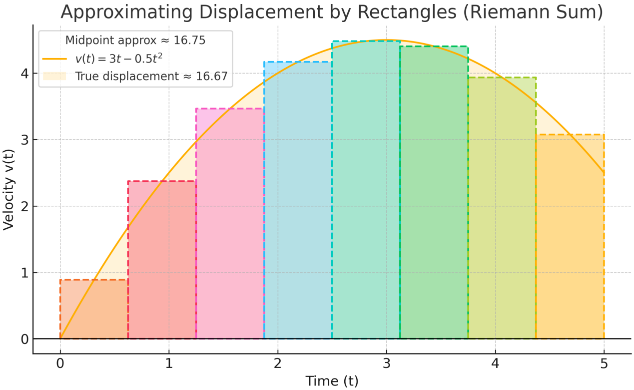


All was light

- Archimedes saw his method as an innovation in engineering rather than mathematics.
- The general rule for calculating such quantities was discovered by Newton 1,700 years later.
- As a physicist, Newton's inspiration came from the relationship between velocity and displacement.
- If an object moves at the velocity of v, how far would it travel in time t?
- What if the velocity is proportional to time, vt?
- And what if the velocity is a function of time, $v(t)$?

All was light

- Newton: The displacement always equals the area under the velocity–time curve!



- Let's denote the displacement at time T by $F(T)$.

All was light

- This observation means $F(T) - F(0) = \sum_{t=0}^{T} v(t)$.
- Summation is not well-defined for continuous time; let's replace it with something similar:

$$F(T) - F(0) = \int_{t=0}^{T} v(t).$$

- Another observation: In a short period Δt, the displacement is approximately $v(t)\Delta t$.
- Therefore, $F'(t) = v(t)$, which gives:

$$F(T) - F(0) = \int_{t=0}^{T} F'(t).$$

- If we can find a function $F(t)$ such that $F'(t) = v(t)$, we solve the problem of calculating the area!
- This is the converse problem of differentiation.

All was light

- Today, we write the formula in a generalized modern form:

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

- This is known as the Newton-Leibniz formula, the cornerstone of calculus.
- It reveals the link between two seemingly unrelated problems: 1) finding the instantaneous rate of change, and 2) calculating the area under a curve.
- With $f(x) = F'(x)$, we call $\int_a^b f(x)dx$ the definite integral of $f(x)$ on $[a, b]$.
- $F(x) + C$ is called the indefinite integral of $f(x)$, where C is any constant.
- The definite integral is a number; the indefinite integral is a function.

All was light

- Summary: To compute any definite integral, we need to find the indefinite integral of $f(x)$, the integrand.
- Newton knew that many functions can be approximated by a sum of polynomials (Taylor expansion).
- For polynomials, the converse problem of differentiation is easy to solve.
- Using this approach, he could find $F(x)$ for a wide range of functions.
- He solved numerous problems this way and was regarded as a magician by his contemporaries.
- One day, Newton received a letter from Germany challenging him with a mathematical problem.
- The sender was a German diplomat, Gottfried Wilhelm Leibniz.

Properties of integration

- Today, mathematicians have computed the derivative for many functions.
- It means we know the solution to many converse problems.
- These results are the foundation of calculating indefinite integrals.
- E.g., $\int x^k dx = \frac{1}{k+1}x^{k+1} + C$, $\int a^x dx = \frac{a^x}{\ln a} + C$, $\int \ln x dx = x \ln x - x + C$, $\int \sin x dx = -\cos x + C$, and $\int \frac{1}{x} dx = \ln |x| + C$.
- The last one holds because when $x < 0$, $\frac{d \ln |x|}{dx} = \frac{d \ln(-x)}{dx} = \frac{1}{x}$.
- Note that $(\int f(x)dx)' = f(x)$ and $\int f'(x)dx = f(x) + C$.
- Integration is also a linear operation:

$$\int (af(x) + bg(x))dx = a \int f(x)dx + b \int g(x)dx.$$

Integration by substitution

- Consider two functions, $y = F(u)$ and $u = g(x)$, and their composite $y = F \circ g(x) = F(g(x))$.
- We learned the chain rule for computing derivatives.

$$(F \circ g(x))' = F'(g(x))g'(x) = f(g(x))g'(x) = h(x).$$

- Integrating on both sides, we get

$$F \circ g(x) = \int f(g(x))g'(x)dx = \int h(x)dx = \int f(u)du + C.$$

- Therefore, if we can find a function $u = g(x)$ such that $h(x) = f(g(x))g'(x)$, we can transform the integration over x to that over u.
- The transformed problem may already have a solution.
- E.g., to compute $\int e^{x^2} x dx$, let $u = x^2$, then

$$\int e^{x^2} x dx = \int e^u d\left(\frac{1}{2}x^2\right) = \int e^u du = e^u = e^{x^2}.$$

Integration by substitution

- Or, we can search for a function $x = m(t)$, such that

$$\int h(x)dx = \int h(m(t))m'(t)dt.$$

- Again, the hope is that $\int h(m(t))m'(t)dt$ is easier to calculate than the original problem.
- It is known as the change of variables formula for integration.
- Nothing but another type of substitution.

Examples

- Let's compute the indefinite integral of $\tan x$:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{d\cos x}{\cos x} = \int -\frac{du}{u}$$
$$= -\ln|u| + C = -\ln|\cos x| + C.$$

- The key is to note that $\sin x dx = -(\cos x)' dx$ and set $u = \cos x$.
- Consider the indefinite integral of $x(2x-1)^{100}$.
- Let $t = 2x - 1$, then $dt = 2dx$, and

$$\int x(2x-1)^{100} dx = \int \frac{t+1}{4} t^{100} dt = \int \frac{t^{101} + t^{100}}{4} dt$$
$$= \frac{1}{4}\left(\frac{t^{102}}{102} + \frac{t^{101}}{101}\right) + C = \frac{1}{4}\left(\frac{(2x-1)^{102}}{102} + \frac{(2x-1)^{101}}{101}\right) + C.$$

Integration by parts

- Remember that

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

- Integrating on both sides, we have

$$\begin{aligned} \int (f(x)g(x))' dx &= f(x)g(x) \\ &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\ &= \int g(x)df(x) + \int f(x)dg(x). \end{aligned}$$

- We often use the following form:

$$\int g(x)df(x) = f(x)g(x) - \int f(x)dg(x).$$

Integration by parts

- It is a very important technique in calculus.
- "The only real-world skill in math is knowing how to change the order of integration."
- By changing the integrand from $g(x)f'(x)$ to $g'(x)f(x)$, it can transform a problem without a solution to a problem with a solution.
- E.g., we don't know the integral of xe^x.
- However, $\int xe^x dx = \int x de^x = xe^x - \int e^x dx = (x-1)e^x$.
- The two techniques can be used together.

Examples

- Let's find the indefinite integral of $\ln x$:

$$\int \ln x dx = x \ln x - \int x d \ln x = x \ln x - x + C.$$

- Compute the indefinite integral of $e^x \sin x$:

$$\int e^x \sin x dx = \int \sin x de^x = e^x \sin x - \int e^x \cos x dx$$
$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx.$$

- Therefore, $\int e^x \sin x dx = \frac{e^x \sin x - e^x \cos x}{2} + C$.

Compute definite integrals

- We can compute the definite integral of any function using the Newton-Leibniz formula.
- The same techniques can be applied here.
- But we should pay attention to the bounds of integration:

$$\int_a^b h(x)dx = \int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

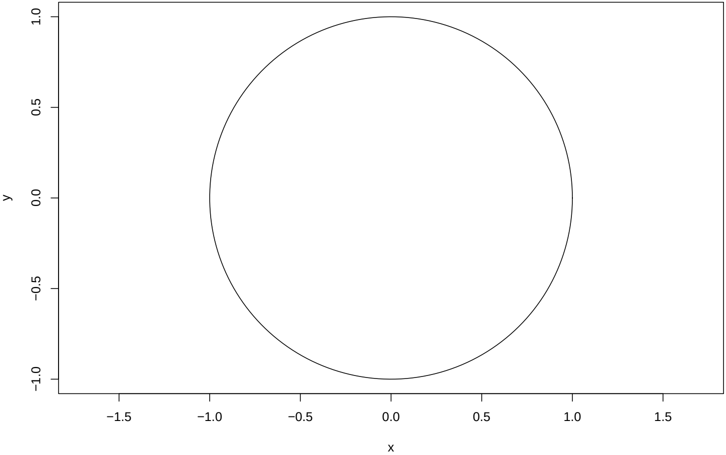
- There are some properties we can rely on:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

- Consequently, $\int_a^a f(x)dx = 0$.
- Definite integrals are directional, thus $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

An application

- We know the formula for the area of a circle is πr^2, where r is the radius.
- A circle can be represented by the equation $x^2 + y^2 = r^2$.



An application

- Consider the case where both x and y are positive.
- Then, the equation can be expressed as a function: $y = \sqrt{r^2 - x^2}$.
- To calculate the area under the curve, we compute the definite integral of the function on $[0, r]$.
- Let $x = r \sin t$, then

$$\int_0^r \sqrt{r^2 - x^2} dx = \int_0^{\pi/2} r\sqrt{r^2 - r^2 \sin^2 t} \cos t dt$$
$$= \int_0^{\pi/2} r^2 \cos^2 t dt = \int_0^{\pi/2} r^2 \cos t d(\sin t)$$
$$= r^2 \sin t \cos t \, |_0^{\pi/2} - \int_0^{\pi/2} r^2 \sin^2 t dt = \int_0^{\pi/2} r^2(1 - \cos^2 t) dt.$$

- Therefore, $\int_0^{\pi/2} r^2 \cos^2 t dt = \frac{r^2}{2} \int_0^{\pi/2} 1 dt = \frac{r^2}{2} \frac{\pi}{2}$.
- The total are is $4 * \frac{r^2}{2} \frac{\pi}{2} = \pi r^2$.

Differentiation of an integral with variable limits

- The Newton-Leibniz formula implies that

$$\frac{d}{dx}\int_{-\infty}^{x} f(t)dt = \frac{d(F(x) - F(-\infty))}{dx} = f(x).$$

- Using the chain rule, we can see

$$\frac{d}{dx}\int_{-\infty}^{g(x)} f(t)dt = f(g(x))g'(x).$$

- Note that $\int_{-\infty}^{g(x)} m(x)f(t)dt = m(x)\int_{-\infty}^{g(x)} f(t)dt$, then

$$\frac{d}{dx}\int_{-\infty}^{g(x)} m(x)f(t)dt = m(x)f(g(x))g'(x) + m'(x)\int_{-\infty}^{g(x)} f(t)dt.$$

Integration for multivariate functions

- For a multivariate function, we can integrate along each of its variables:

$$\int_a^b \int_c^d f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dx \right) dy.$$

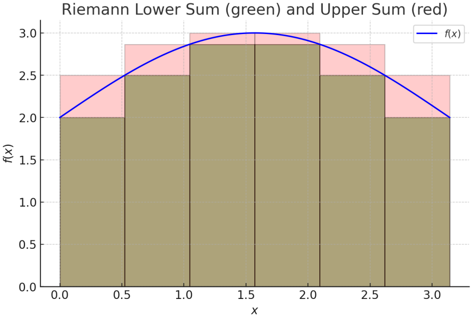
- The inside integral gives a function of y.
- We can integrate along a curve or a surface.
- Each of our organs (e.g., heart) can be described by a function $f(x,y)$.
- Each CT scan can be seen as its integral along a line.
- Can we reconstruct $f(x,y)$ from its integrals along all possible lines?
- The answer is positive (Radon Transform).

Leibniz's insight

- Leibniz was the first to note the power of first-order approximation.
- Consider $(2.001)^3 = 8.012006001 = 8 + 0.012 + 0.000006 + 0.0\ldots01$.
- What really matters in practice is 0.012; the remaining terms are negligible.
- Therefore, $\Delta y \approx f'(x)\Delta x$, or simply $dy = f'(x)dx$.
- Consider the area A under the curve $f(x)$.
- What is the change in A if x increases by a small amount?
- The answer can be derived from the rectangular's area: $dA = f(x)dx$.
- Therefore, $f(x) = \frac{dA}{dx}$ and $A = \int f(x)dx$.
- We get the fundamental theorem of calculus from a different angle.

What is an integral?

- Integral was widely used in practice since it was invented.
- But the logical foundation was not solid as it relies on ambiguous concepts such as infinitesimals.
- The rigorous definition based on limits was given by Bernhard Riemann.
- He defines the definite integral as the limit of two sequences of sums (upper sum and lower sum).



What is an integral?

- But what functions are (Riemann) integrable?
- It leads to a more fundamental problem: what is area?
- Can we define the area for any set of points on a plane?
- Henri Lebesgue: not really.
- He developed the measure theory and extended the definition of integrals.
- Consider the Dirichlet function where p and q are integers:

$$f(x) = \begin{cases} 1 & x = \frac{p}{q}, \\ 0 & \text{otherwise.} \end{cases}$$

- What is its definite integral on $[0, 1]$?
- The upper sum is always 1 while the lower sum is always 0.
- By Riemann's definition, this function doesn't have a definite integral on $[0, 1]$.

What is an integral?

- Henri Lebesgue: maybe we just need to modify the definition.
- Suppose you have some quarters and a lot of cents in your pocket.
- How do you know how much money you have?
- Riemann: count every coin one by one and add up their values.
- Lebesgue: group coins by denomination first, then sum up within each group.
- For the Dirichlet function, we integrate on rational and irrational numbers separately.
- There are infinitely many irrationals, but the function's value is 0 there.
- On rationals, the function's value is 1, but rationals form a set of measure zero.
- Consequently, the (Lebesgue) integral of the function on $[0, 1]$ is 0.