Limits

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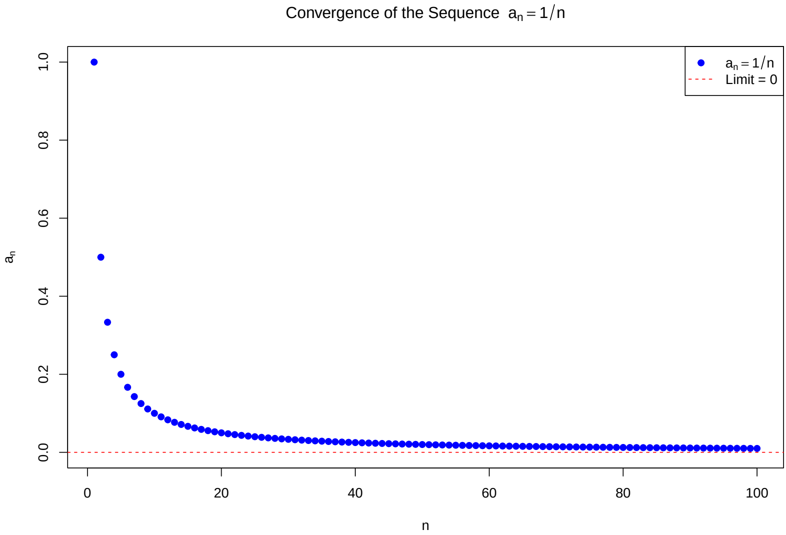
Sequence

- A sequence is an ordered list of numbers.
- Numbers in a sequence can be indexed by positive integers: $\{a_n\}_{n=1}^{\infty}$.
- Consider all numbers in a sequence as a set.
- There exists a function from $\mathbb{N}$ to this set: $f(n) = a_n$.
- The function $f()$ is called the general term of the sequence.
- E.g., for the sequence $\{1, 3, 5, 7, \dots\}$, what is the form of $f()$?
- The cardinality of this set is at most countably infinite.
- We focus on infinite sequences.

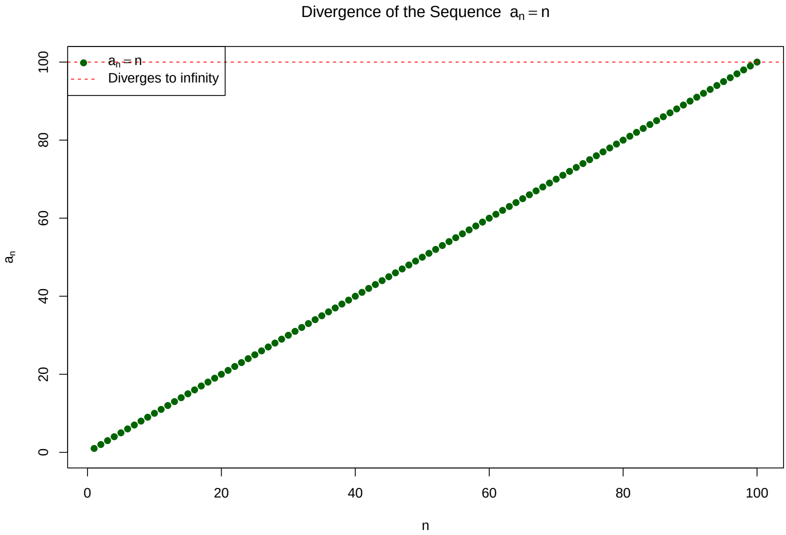
Limit of a sequence

- We are often interested in the sequence's value when $n \to \infty$.
- E.g., what is China's economic growth in the long run?
- We call this quantity the limit of this sequence: $\lim_{n\to\infty} a_n$.
- Not all sequences have a finite limit.
- E.g., as $n \to \infty$, $2^n \to \infty$.
- A sequence is convergent if it has a finite limit; otherwise it is divergent.
- A sequence is called a null sequence or a vanishing sequence if its limit is zero.

Limit of a sequence



Limit of a sequence



Series

- The sum of the first n terms of a sequence defines a new sequence:

$$S_n = \sum_{i=1}^{n} a_i.$$

- The limit of S_n is called a series.
- A convergent sequence may not result in a convergent series.
- E.g., $a_n = \frac{1}{n} \to 0$ as $n \to \infty$ but $\sum_{i=1}^{n} \frac{1}{i} \to \infty$.
- We can first calculate S_n, and then investigate its behavior when n grows.

An example

- In formal theory, we often consider players facing an infinite horizon.
- For instance, suppose an autocrat receives a payoff of b_a in each period.
- Future payoffs are worth less to him due to discounting.
- A payoff received t periods from now is discounted by a factor of δ^t, where $0 < \delta < 1$.
- Therefore, the autocrat's stream of future payoffs forms a sequence: $\{a_t\}_{t=1}^{\infty} = \{b_a \delta^t\}_{t=1}^{\infty}$.
- What he cares about is the total discounted value of these future payoffs: $\sum_{t=1}^{\infty} b_a \delta^t$.
- It is known that $\sum_{t=1}^{T} b_a \delta^t = b_a \frac{1-\delta^T}{1-\delta}$.
- As $T \to \infty$, $\delta^T \to 0$, and the total discounted value converges to $\frac{b_a}{1-\delta}$.

An example

- Suppose a revolution breaks out, and the autocrat must decide whether to repress it.
- Repression incurs a cost c, but it always succeeds.
- If the autocrat chooses not to repress, democratization occurs, and his payoff becomes b_d in each subsequent period.
- Thus, his expected payoff is $\frac{b_a}{1-\delta} - c$ if he represses, and $\frac{b_d}{1-\delta}$ if he does not.
- When will the autocrat choose to repress?
- He will do so if and only if $b_a - b_d \geq c(1-\delta)$.
- This inequality forms the foundation of many formal models of political transition.

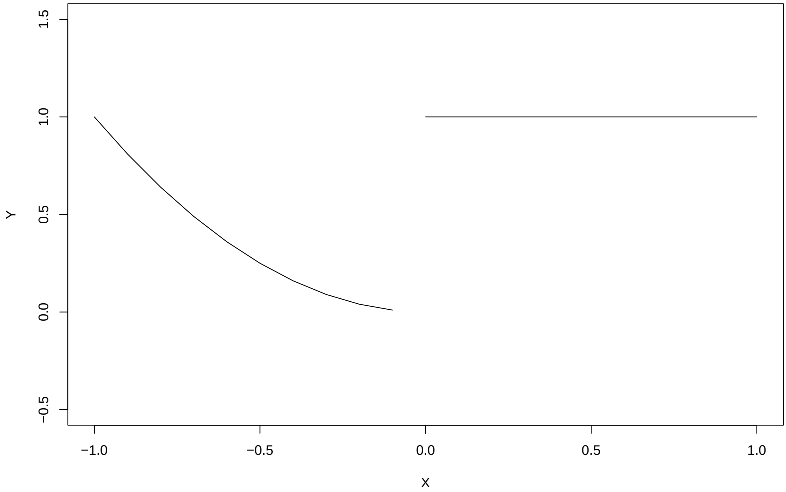
Limits of functions

- We can generalize the definition of limits to all functions.
- What value does $f(x)$ approach when $x \to x_0$?
- This value is known as $f(x)$'s limit as $x \to x_0$; we denote it as $\lim_{x \to x_0} f(x)$.
- The answer is not always $f(x_0)$.
- Consider the following function:

$$Y = \begin{cases} 1 & \text{if } X \geq 0 \\ X^2 & \text{if } X < 0. \end{cases}$$

- We know that $f(0) = 1$.
- What about $\lim_{x \to 0} f(x)$?

Limits of functions



Continuous functions

- If we let $x \to 0$ from the left side, $f(x) \to 0$.
- If we let $x \to 0$ from the right side, $f(x) \to 1$.
- We call 0 the left limit of $f(x)$: $\lim_{x \to 0-} f(x) = 0$.
- Similarly, 1 the right limit of $f(x)$: $\lim_{x \to 0+} f(x) = 1$.
- We say $f(x)$ is left-continuous at x_0 if $\lim_{x \to x_0-} f(x) = f(x_0)$ and $f(x)$ is right-continuous at x_0 if $\lim_{x \to x_0+} f(x) = f(x_0)$.
- $f(x)$ is continuous at x_0 if $\lim_{x \to x_0-} f(x) = \lim_{x \to x_0+} f(x) = f(x_0)$.
- Continuity is a point-wise property.
- If $f(x)$ is continuous at every point in its domain, then $f(x)$ is a continuous function.

Continuous functions

- If $f(x)$ is continuous at x_0, then we can infer the value of $f(x_0)$ by examining values of $f(x)$ near x_0.
- E.g., suppose we want to estimate the probability that individuals with a monthly income of 5k support Trump.
- In our survey data, no respondent reports an exact income of 5k.
- However, assuming that support for Trump varies continuously with income, we can approximate the value by averaging the responses from individuals whose monthly incomes fall between 4.5k and 5.5k.
- There exists a connection between limits of sequences and the continuity of a function at x_0.
- $f(x)$ is continuous at x_0 if and only if for any sequence $\{x_n\}_{n=1}^{\infty} \to x_0$, $\lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n)$.

Limits: rigorous definition (*)

- In mathematics, limits are defined rigorously using the $\epsilon - \delta$ formalism.
- $\lim_{x \to x_0} f(x) = y$ means the following: for any $\epsilon > 0$, we can find a $\delta > 0$ such that for any $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$.
- Similarly, $\lim_{n \to \infty} a_n = a$ means: for any $\epsilon > 0$, we can find some $N > 0$, such that for any $n > N$, $|a_n - a| < \epsilon$.
- This formalism is quite useful in proofs.
- E.g., if $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$, then $\lim_{n \to \infty}(a_n + b_n) = a + b$.
- By definition, for any $\epsilon/2 > 0$, we can find some $N_a, N_b > 0$, such that for any $n > N_a$, $|a_n - a| < \epsilon/2$, and for any $n > N_b$, $|b_n - b| < \epsilon/2$.
- Then, taking $N = \max\{N_a, N_b\}$, we know that for any $n > N$, $|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \epsilon$.
- We can similarly prove that $\lim_{n \to \infty} a_n b_n = ab$.

Open and closed sets

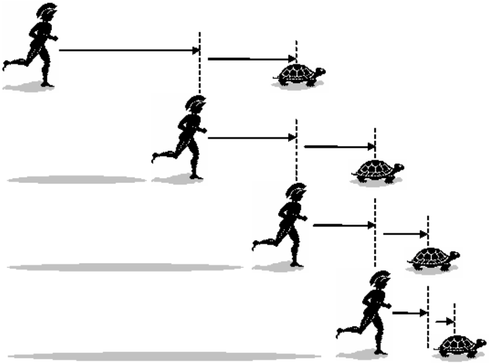
- Using the concept of limits, we can discuss the topology of sets.
- A set S is an open set if for any $x \in S$, some neighborhood of x is also in S.
- More formally, if for any $x \in S$, we can find $\delta > 0$ such that for any $|x' - x| < \delta$, $x' \in S$, then S is an open set.
- An open set cannot have any boundary point (Why?).
- E.g., a voter's policy preferences are often depicted as an open set.
- A closed set S contains all its limit points.
- More formally, if for any $\{x_n\}_{n=1}^{\infty} \in S$, where $\lim_{n\to\infty} x_n = x$, $x \in S$, then S is a closed set.
- If S is closed and bounded, we say S is compact.
- These concepts are central for optimization problems.
- Common examples: an open interval (a, b) and a closed interval $[a, b]$.
- The complement of an open (closed) set is a closed (open) set.

Limits: a history

- Zeno's paradox: Can Achilles ever catch up with the tortoise?
- Suppose Achilles starts 100 meters behind the tortoise.
- Achilles runs at 10 meters per second; the tortoise at 1 meter per second.
- When will Achilles catch up with the tortoise?

Limits: a history

- Zeno: for Achilles to catch the tortoise, he must first reach the spot where the tortoise was.
- But by the time Achilles gets there, the tortoise has already moved a little farther.
- If we repeat this reasoning infinitely, it seems Achilles will never catch up.



Limits: a history

- So what's really going on here?
- Let's track the distance between Achilles and the tortoise each time Achilles reaches the tortoise's previous position.
- In round 1, the distance is 10 meters; in round 2, it's 1 meter; in round 3, 0.1 meter; and so on.
- This gives us a sequence: $\{10, 1, 0.1, \ldots, 10 * 0.1^{n-1}, \ldots\}$.
- The distances shrink with each round, and the time Achilles takes also gets shorter and shorter.
- This seemingly endless process actually adds up to a finite time

Limits: a history

- The concept of limits was not clear even after calculus was invented.
- Newton used the expression of “infinitesimal” arbitrarily.
- Sometimes it means zero, and other times it is a really smaller number.
- The confusion persisted until the 18th Century.
- Augustin-Louis Cauchy was the first to define limits rigorously in calculus.
- He replaced “infinitesimal changes” with the idea of approaching a value.
- Karl Weierstrass formalized limits using precise logic and introduced the $\epsilon - \delta$ formalism into math.