Sets and Functions

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Sets

- Set theory is the foundation of modern mathematics.
- A set S is a collection of whatever elements.
- Whatever you want to study can be included in a set.
- E.g., $\{42\}$, $\{Ye, Jaime\}$, $\{0, 1, 2, 3, \dots\}$.
- We say $x \in S$ if an element x belongs to S.
- We often define a set using conditions:
  $S = \{x : x \text{ is a US senator}\}$.
- Some sets we will often use: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.
- The number of elements in S is called its cardinality, $|S|$.
- It could be finite or infinite.

Sets

- If $|S| = 0$: empty set; if $|S| = 1$: singleton; if $|S| = |\mathbb{N}|$: countable set; if $|S| > |\mathbb{N}|$: uncountable set.
- Is $\mathbb{Q}$ a countable set? How about $\mathbb{R}$?
- We can prove such propositions using Cantor's diagonal argument.
- Hilbert's first problem (continuum hypothesis): Is there a set $\mathbb{S}$ such that $|\mathbb{N}| < |\mathbb{S}| < |\mathbb{R}|$?
- Kurt Godel and Paul Cohen: it cannot be proved under the popular axiomatic system.
- S_1 is a subset of S if for any $x \in S_1$, $x \in S$.
- We denote it as $S_1 \subset S$.
- The collection of all subsets of a set S is called the power set and denoted as 2^S.
- What is the power set of $\{Ye, Jaime\}$?

Operations on sets

- We can conduct operations on sets.
- Consider two sets A and B.
- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- Complement: $A^c = \{x : x \notin A\}$.
- Difference: $A \setminus B = \{x : x \in A \text{ and } x \notin A \cap B\}$.
- A useful tool: Venn diagram.

Operations on sets

- From their definitions, we can see the connection between set operations and logical operations.
- Union - OR; Intersection - AND; Complement - NOT; Subset - IMPLIES.
- Let's look at several properties of set operations.
- Commutative Laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- Associative Laws: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
- Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- De Morgan's laws: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Operations on sets

- Consider an example.
- $A = \{\text{Qualtrics}, \text{Google Forms}, \text{Excel}\}$: tools for data collection; $B = \{\text{Excel}, \text{R}, \text{Python}, \text{Stata}\}$: tools for data analysis; $C = \{\text{Word}, \text{R}, \text{Latex}\}$: tools for writing.
- The universal set is

$$\{\text{Qualtrics}, \text{Google Forms}, \text{Excel}, \text{R}, \text{Python}, \text{Stata}, \text{Word}, \text{Latex}\}.$$

- Apply these laws to the three sets.

Function: definition

- The relationship between two sets can be represented by a function/mapping.
- For two sets A and B, a function $f()$ assigns each $x \in A$ to a unique $f(x) = y \in B$.
- We denote it as $f : A \to B$.
- A is the domain of f, while B is the codomain of f.
- y is x's image in B, and x is y's inverse image.
- The collection of each x's image is the range of $f()$.
- The range is a subset of the codomain.

Common functions

- Linear function: $y = a + bx$.
- Quadratic function: $y = ax^2 + bx + c$.
- Power function: $y = ax^b$.
- Exponential function: $y = e^x$.
- Logarithmic function: $y = \log(x)$.
- Trigonometric functions: $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$.
- What are the domain and range for each function?

Composite function

- We can apply a function to the output of another function.
- E.g., $x \to a + bx \to e^{a+bx}$.
- $f_1(x) = a + bx$, $f_2(a + bx) = e^{a+bx}$.
- We write it as $f_1 \circ f_2(x) = e^{a+bx}$.
- $f_1 \circ f_2(x)$ is called a composite function.
- E.g., voter's preference - votes - electoral results.

Bijective

- We say f is surjective if any $y \in B$ has an inverse image in A.
- f is injective if each element in the range has one inverse image in A.
- f is bijective if it is injective and surjective.
- In the following examples, is the function injective, surjective, or bijective?
  - Voter ID - Voter profile
  - Candidate ID - Candidate name
  - Department - University

Inverse function

- For a bijective function, we can define its inverse function $f^{-1}()$:
$$\text{If } f(x) = y, \text{ then } f^{-1}(y) = x.$$
- The inverse function cancels whatever the function does.
- It means that $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$.
- E.g., $e^{\log(x)} = \log(e^x) = x$.

Example I: production function

- In economics, scholars use production functions to describe the relationship between input and output in production.
- Typically, we assume that the product y is a function of two variables, labor (L) and capital (K).
- The function can be expressed as $y = f(K, L)$.
- Linear production function: $y = aK + bL$.
- Leontief production function: $y = \min\{aK, bL\}$.
- Cobb-Douglas production function: $y = AK^b L^c$.
- Equivalently, $\log y = \log A + b \log K + c \log L$.

Example II: electoral system

- An electoral system can be seen as a function.
- It maps vote shares of parties/candidates to the electoral result.
- Let's consider several systems: plurality, PR, two-round runoff, Borda count.
- How do we express them as functions?
- Are they injective, surjective, or bijective?